Russian mathematician announces proof of celebrated Poincaré Conjecture

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In early April 2002, Dr. Grigori Perelman of the Steklov Institute of Mathematics in St. Petersburg gave a series of public lectures at the Massachusetts Institute of Technology. In the lectures he explained work laid out in two articles, and how this work will establish a number of important mathematical results, including the famous Poincaré Conjecture. Mathematicians are still examining Perelman's arguments for possible errors, but up to now they have withstood all criticism.[1]

[In considering the following explanation, we advise readers to either locate actual ball and doughnut shapes to look at, or to use pencil and paper to draw them. This makes visualizing and grasping the content of this article easier.]

Poincaré's conjecture and Perelman's work deal with mathematical objects called manifolds. Loosely speaking, these are geometric objects that "up close" look like a line segment (one-dimensional manifolds), a disk in the plane (two-dimensional manifolds), a ball in solid space (three-dimensional manifolds), and so on in higher dimensions.[2]

The surface of a ball is an example of a two-dimensional manifold: to a very small ant walking along the surface of a very large beach ball, it always appears that he is walking on a flat disk. The fact that the surface of the earth is a two-manifold, and hence "up close" looks like a plane, made early humanity theorize that the earth was flat. However, pictures of the earth taken from space show that the surface of the earth is not a flat plane, but actually is also the surface of a ball.

The previous two examples give rise to a very important idea of equivalence. If one had a (very stretchy and malleable) beach ball and a lot of air, one could imagine inflating it, stretching it and pulling it so that it actually took on the shape of the surface of the earth. Mathematicians express this by saying that the surface of a beach ball and that of the earth are *topologically equivalent*.

However, not all surfaces are topologically equivalent: for example, one may compare the surface of a ball and the surface of a doughnut. One observes that a loop on the surface of a ball (the 2-dimensional sphere) can always be pulled back along the surface until it collapses to a point. However, a loop around the inner hole of the surface of a doughnut cannot be shrunk to a point without actually cutting into the doughnut or otherwise leaving its surface. Poincaré showed that this meant that the surface of the ball and that of a doughnut could not be topologically equivalent. In fact, a beautiful classification theorem known to mathematicians of Poincaré's time shows that any surface on which all loops can be shrunk down to a point is topologically equivalent to the 2-dimensional sphere.

The Poincaré Conjecture tries to generalize this to higher dimensions. Specifically, it asks: is every 3-dimensional manifold with the property that all loops on it can be shrunk to a point topologically equivalent to the 3-dimensional sphere?

Explaining precisely what a 3-dimensional sphere is to a lay audience presents some difficulties, as it is harder to visualize. Technically, one reasons by analogy. The fact that one traces out a 1-dimensional sphere (the edge of a circle) with a compass on a 2-dimensional plane indicates that a 1-dimensional sphere consists of the points in 2-dimensional space a fixed distance away from given point (the needle point of the compass). Similarly, the 2-dimensional sphere (the surface of a solid ball) consists of the points a fixed distance away from a given point in 3-dimensional

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space. So, the 3-dimensional sphere consists of the points a fixed distance away from a given point in 4-dimensional space.

The Poincaré Conjecture remained unsolved during the entire twentieth century and defeated the efforts of many of the best topologists and geometers of the time. It acquired a status in the mathematical world similar to that of Fermat's famous Last Theorem, recently proved by Andrew Wiles. By the mid-twentieth century analogous versions of the Poincaré Conjecture had been shown to be true in dimensions above 3. However, all of the numerous efforts on the three-dimensional case failed.

Perelman's work aims to prove the Poincaré Conjecture by proving a far larger classification theorem, the recent Geometrization Conjecture of William Thurston. Thurston's Geometrization Conjecture predicts that any 3-manifold can be cut up into pieces, each of which can be stretched and bent until it possesses one of eight fixed geometric structures.

The study of these geometric structures is differential geometry—the basic mathematical language of Einstein's general relativity theory in physics, and Perelman's area of expertise. Broadly speaking, a geometric structure on a manifold is a way of specifying the behavior of shortest paths between pairs of points in the manifold.

We will give only one example. On the surface of the earth, the shortest path between two points (taking one of the points to be the North Pole) is along the meridian of fixed longitude connecting the North Pole to the other point. This is why if one draws the flight path of a New York-Tokyo flight on a flat world map, it does not follow the straight line connecting the two cities on the map, but instead curves up north over Canada and then down along the coast of Northeast Asia. The airplane is roughly following the (curved) minimum-length path between New York and Tokyo on the surface of the earth, the famous "great circle" route.

Much of the work of differential geometry over the last century has been to establish links between the topological properties of manifolds (e.g. the structure of loops on them) and what sort of geometric structures they can have. If Perelman's work does give a proof of Thurston's Geometrization Conjecture, this together with previous work will establish that if a 3-dimensional manifold has all of its loops shrinkable to points, it carries a geometric structure that forces it to be topologically equivalent to the three-dimensional sphere, proving Poincaré's Conjecture.

Perelman's spectacular efforts towards solving several of the great problems in three-dimensional geometry are particularly remarkable since they are taking place in a mathematical environment devastated by the collapse of the Soviet Union. The economic "shock therapy" applied to the former USSR forced universities across the country to suspend payment of professors' salaries, and resulted, during the mid-1990s, in a massive flight of trained mathematicians from the former USSR to universities in the wealthy countries, especially the US.

Notes:

1. Perelman's articles are highly technical and written for specialists in the field of differential geometry. However, they are publicly available online at the arxiv server, which mathematicians now commonly use to post their results. Interested readers may consult them at http://www.arxiv.org/abs/math.DG/0211159 and http://www.arxiv.org/abs/math.DG/0303109.
2. There is a technical disclaimer: to exclude bad behavior, we will consider only manifolds that are compact—roughly speaking, don't stretch out forever—and connected—i.e., come in one piece. The other manifolds are fairly easy to obtain from these.

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